Exercise 11

Convert each of the following Fredholm integral equation in 9–16 to an equivalent BVP:

$$u(x) = \cos x + \int_0^1 K(x,t)u(t)\,dt,\; K(x,t) = \begin{cases} 6t(1-x), & \text{for } 0 \le t \le x \\ 6x(1-t), & \text{for } x \le t \le 1 \end{cases}$$

Solution

Substitute the given kernel $K(x,t)$ into the integral.

$$u(x) = \cos x + \int_0^x 6t(1-x)u(t)\,dt + \int_x^1 6x(1-t)u(t)\,dt \tag{1}$$

Differentiate both sides with respect to x.

$$u'(x) = -\sin x + \frac{d}{dx}\int_0^x 6t(1-x)u(t)\,dt + \frac{d}{dx}\int_x^1 6x(1-t)u(t)\,dt$$

Apply the Leibniz rule to differentiate the integrals.

$$\begin{aligned}
&= -\sin x + \int_0^x \frac{\partial}{\partial x}6t(1-x)u(t)\,dt + \cancel{6x(1-x)u(x)\cdot 1} - 6(0)(1-x)u(0)\cdot 0 \\
&\qquad + \int_x^1 \frac{\partial}{\partial x}6x(1-t)u(t)\,dt + 6x(0)u(1)\cdot 0 - \cancel{6x(1-x)u(x)\cdot 1} \\
&= -\sin x + \int_0^x (-6t)u(t)\,dt + \int_x^1 6(1-t)u(t)\,dt \\
&= -\sin x - 6\int_0^x tu(t)\,dt - 6\int_1^x (1-t)u(t)\,dt
\end{aligned}$$

Differentiate both sides with respect to x once more.

$$\begin{aligned}
u''(x) &= -\cos x - 6\frac{d}{dx}\int_0^x tu(t)\,dt - 6\frac{d}{dx}\int_1^x (1-t)u(t)\,dt \\
&= -\cos x - 6xu(x) - 6(1-x)u(x) \\
&= -\cos x - \cancel{6xu(x)} - 6u(x) + \cancel{6xu(x)}
\end{aligned}$$

The boundary conditions are found by setting $x = 0$ and $x = 1$ in equation (1).

$$\begin{aligned}
u(0) &= \cos 0 + \int_0^0 6t(1)u(t)\,dt + \int_0^1 6(0)(1-t)u(t)\,dt = 1 \\
u(1) &= \cos 1 + \int_0^1 6t(0)u(t)\,dt + \int_1^1 6(1)(1-t)u(t)\,dt = \cos 1
\end{aligned}$$

Therefore, the equivalent BVP is

$$u'' + 6u = -\cos x,\; u(0) = 1,\; u(1) = \cos 1.$$