Exercise 12

Convert each of the following Fredholm integral equation in 9–16 to an equivalent BVP:

$$u(x) = \sinh x + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 4t(1-x), & \text{for } 0 \le t \le x \\ 4x(1-t), & \text{for } x \le t \le 1 \end{cases}$$

Solution

Substitute the given kernel $K(x,t)$ into the integral.

$$u(x) = \sinh x + \int_0^x 4t(1-x)u(t)\,dt + \int_x^1 4x(1-t)u(t)\,dt \tag{1}$$

Differentiate both sides with respect to x.

$$u'(x) = \cosh x + \frac{d}{dx}\int_0^x 4t(1-x)u(t)\,dt + \frac{d}{dx}\int_x^1 4x(1-t)u(t)\,dt$$

Apply the Leibniz rule to differentiate the integrals.

$$\begin{aligned} &= \cosh x + \int_0^x \frac{\partial}{\partial x} 4t(1-x)u(t)\,dt + \cancel{4x(1-x)u(x)\cdot 1} - 4(0)(1-x)u(0)\cdot 0 \\ &\qquad + \int_x^1 \frac{\partial}{\partial x} 4x(1-t)u(t)\,dt + 4x(0)u(1)\cdot 0 - \cancel{4x(1-x)u(x)\cdot 1} \\ &= \cosh x + \int_0^x (-4t)u(t)\,dt + \int_x^1 4(1-t)u(t)\,dt \\ &= \cosh x - 4\int_0^x tu(t)\,dt - 4\int_1^x (1-t)u(t)\,dt \end{aligned}$$

Differentiate both sides with respect to x once more.

$$\begin{aligned} u''(x) &= \sinh x - 4\frac{d}{dx}\int_0^x tu(t)\,dt - 4\frac{d}{dx}\int_1^x (1-t)u(t)\,dt \\ &= \sinh x - 4xu(x) - 4(1-x)u(x) \\ &= \sinh x - \cancel{4xu(x)} - 4u(x) + \cancel{4xu(x)} \end{aligned}$$

The boundary conditions are found by setting $x = 0$ and $x = 1$ in equation (1).

$$\begin{aligned} u(0) &= \sinh 0 + \int_0^0 4t(1)u(t)\,dt + \int_0^1 4(0)(1-t)u(t)\,dt = 0 \\ u(1) &= \sinh 1 + \int_0^1 4t(0)u(t)\,dt + \int_1^1 4(1)(1-t)u(t)\,dt = \sinh 1 \end{aligned}$$

Therefore, the equivalent BVP is

$$u'' + 4u = \sinh x,\ u(0) = 0,\ u(1) = \sinh 1.$$