Exercise 3

Convert each of the following BVPs in 1–8 to an equivalent Fredholm integral equation:

$$y'' + 2y = x,\ 0 < x < 1,\ y(0) = 1,\ y(1) = 0$$

Solution

Let

$$y''(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x.

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$
$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

Bring $y'(0)$ to the right side.

$$y'(x) = y'(0) + \int_0^x u(t)\,dt$$

Integrate both sides from 0 to x again.

$$\int_0^x y'(r)\,dr = \int_0^x \left[y'(0) + \int_0^r u(t)\,dt\right] dr$$
$$y(x) - y(0) = y'(0)x + \int_0^x \int_0^r u(t)\,dt\,dr$$

Substitute $y(0) = 1$ and bring it to the right side.

$$y(x) = 1 + y'(0)x + \int_0^x \int_0^r u(t)\,dt\,dr$$

Use integration by parts to write the double integral as a single integral. Let

$$v = \int_0^r u(t)\,dt \qquad dw = dr$$
$$dv = u(r)\,dr \qquad w = r$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned}
y(x) &= 1 + y'(0)x + r\int_0^r u(t)\,dt\bigg|_0^x - \int_0^x ru(r)\,dr \\
&= 1 + y'(0)x + x\int_0^x u(t)\,dt - \int_0^x ru(r)\,dr \\
&= 1 + y'(0)x + x\int_0^x u(t)\,dt - \int_0^x tu(t)\,dt \\
&= 1 + y'(0)x + \int_0^x (x-t)u(t)\,dt
\end{aligned}$$

In order to determine $y'(0)$, set $x = 1$ in this equation for $y(x)$.

$$y(1) = 1 + y'(0) + \int_0^1 (1-t)u(t)\,dt$$

Substitute $y(1) = 0$ and solve for $y'(0)$.

$$0 = 1 + y'(0) + \int_0^1 (1-t)u(t)\,dt \quad \rightarrow \quad y'(0) = -1 - \int_0^1 (1-t)u(t)\,dt$$

Plug this result for $y'(0)$ back into the formula for $y(x)$.

$$y(x) = 1 + x\left[-1 - \int_0^1 (1-t)u(t)\,dt\right] + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = 1 - x - x\int_0^1 (1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt \tag{2}$$

Now plug equations (1) and (2) into the original ODE.

$$y'' + 2y = x \quad \rightarrow \quad u(x) + 2\left[1 - x - x\int_0^1 (1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt\right] = x$$

Expand the left side.

$$u(x) + 2 - 2x - 2x\int_0^1 (1-t)u(t)\,dt + 2\int_0^x (x-t)u(t)\,dt = x$$

Solve for $u(x)$.

$$\begin{aligned}
u(x) &= -2 + 3x + 2x\int_0^1 (1-t)u(t)\,dt - 2\int_0^x (x-t)u(t)\,dt \\
&= -2 + 3x + \int_0^1 2x(1-t)u(t)\,dt - \int_0^x 2(x-t)u(t)\,dt \\
&= -2 + 3x + \int_0^x 2x(1-t)u(t)\,dt + \int_x^1 2x(1-t)u(t)\,dt - \int_0^x 2(x-t)u(t)\,dt \\
&= -2 + 3x + \int_0^x [2x(1-t) - 2(x-t)]u(t)\,dt + \int_x^1 2x(1-t)u(t)\,dt \\
&= -2 + 3x + \int_0^x (-2xt + 2t)u(t)\,dt + \int_x^1 2x(1-t)u(t)\,dt \\
&= -2 + 3x + \int_0^x 2t(1-x)u(t)\,dt + \int_x^1 2x(1-t)u(t)\,dt
\end{aligned}$$

Therefore, the equivalent Fredholm integral equation is

$$u(x) = -2 + 3x + \int_0^1 K(x,t)u(t)\,dt,$$

where

$$K(x,t) = \begin{cases} 2t(1-x) & 0 \le t \le x \\ 2x(1-t) & x \le t \le 1 \end{cases}.$$