Exercise 4

Convert each of the following BVPs in 1–8 to an equivalent Fredholm integral equation:

$$y'' + 3xy = 4, \ 0 < x < 1, \ y(0) = 0, \ y(1) = 0$$

[TYPO: To get the answer at the back of the book, use $y(1) = 1$ instead.]

Solution

Let

$$y''(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x.

$$\begin{aligned} \int_0^x y''(t)\,dt &= \int_0^x u(t)\,dt \\ y'(x) - y'(0) &= \int_0^x u(t)\,dt \end{aligned}$$

Bring $y'(0)$ to the right side.

$$y'(x) = y'(0) + \int_0^x u(t)\,dt$$

Integrate both sides from 0 to x again.

$$\begin{aligned} \int_0^x y'(r)\,dr &= \int_0^x \left[y'(0) + \int_0^r u(t)\,dt\right] dr \\ y(x) - y(0) &= y'(0)x + \int_0^x \int_0^r u(t)\,dt\,dr \end{aligned}$$

Substitute $y(0) = 0$.

$$y(x) = y'(0)x + \int_0^x \int_0^r u(t)\,dt\,dr$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned} v &= \int_0^r u(t)\,dt & dw &= dr \\ dv &= u(r)\,dr & w &= r \end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned} y(x) &= y'(0)x + r\int_0^r u(t)\,dt\bigg|_0^x - \int_0^x ru(r)\,dr \\ &= y'(0)x + x\int_0^x u(t)\,dt - \int_0^x ru(r)\,dr \\ &= y'(0)x + x\int_0^x u(t)\,dt - \int_0^x tu(t)\,dt \\ &= y'(0)x + \int_0^x (x-t)u(t)\,dt \end{aligned}$$

In order to determine $y'(0)$, set $x = 1$ in this equation for $y(x)$.

$$y(1) = y'(0) + \int_0^1 (1-t)u(t)\,dt$$

Substitute $y(1) = 1$ and solve for $y'(0)$.

$$1 = y'(0) + \int_0^1 (1-t)u(t)\,dt \quad \rightarrow \quad y'(0) = 1 - \int_0^1 (1-t)u(t)\,dt$$

Plug this result for $y'(0)$ back into the formula for $y(x)$.

$$y(x) = x\left[1 - \int_0^1 (1-t)u(t)\,dt\right] + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = x - x\int_0^1 (1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt \tag{2}$$

Now plug equations (1) and (2) into the original ODE.

$$y'' + 3xy = 4 \quad \rightarrow \quad u(x) + 3x\left[x - x\int_0^1 (1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt\right] = 4$$

Expand the left side.

$$u(x) + 3x^2 - 3x^2\int_0^1 (1-t)u(t)\,dt + 3x\int_0^x (x-t)u(t)\,dt = 4$$

Solve for $u(x)$.

$$\begin{aligned}
u(x) &= 4 - 3x^2 + 3x^2\int_0^1 (1-t)u(t)\,dt - 3x\int_0^x (x-t)u(t)\,dt \\
&= 4 - 3x^2 + \int_0^1 3x^2(1-t)u(t)\,dt - \int_0^x 3x(x-t)u(t)\,dt \\
&= 4 - 3x^2 + \int_0^x 3x^2(1-t)u(t)\,dt + \int_x^1 3x^2(1-t)u(t)\,dt - \int_0^x 3x(x-t)u(t)\,dt \\
&= 4 - 3x^2 + \int_0^x [3x^2(1-t) - 3x(x-t)]u(t)\,dt + \int_x^1 3x^2(1-t)u(t)\,dt \\
&= 4 - 3x^2 + \int_0^x (-3x^2t + 3xt)u(t)\,dt + \int_x^1 3x^2(1-t)u(t)\,dt \\
&= 4 - 3x^2 + \int_0^x 3xt(1-x)u(t)\,dt + \int_x^1 3x^2(1-t)u(t)\,dt
\end{aligned}$$

Therefore, the equivalent Fredholm integral equation is

$$u(x) = 4 - 3x^2 + \int_0^1 K(x,t)u(t)\,dt,$$

where

$$K(x,t) = \begin{cases} 3xt(1-x) & 0 \le t \le x \\ 3x^2(1-t) & x \le t \le 1 \end{cases}.$$