

Exercise 7

Convert each of the following BVPs in 1–8 to an equivalent Fredholm integral equation:

$$y'' + 4y = x, \quad 0 < x < 1, \quad y(0) = 1, \quad y'(1) = 0$$

Solution

Let

$$y''(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x .

$$\begin{aligned} \int_0^x y''(t) dt &= \int_0^x u(t) dt \\ y'(x) - y'(0) &= \int_0^x u(t) dt \\ y'(x) &= y'(0) + \int_0^x u(t) dt \end{aligned}$$

In order to determine $y'(0)$, set $x = 1$ in this equation for $y'(x)$.

$$y'(1) = y'(0) + \int_0^1 u(t) dt$$

Substitute $y'(1) = 0$ and solve for $y'(0)$.

$$0 = y'(0) + \int_0^1 u(t) dt \quad \rightarrow \quad y'(0) = - \int_0^1 u(t) dt$$

Substitute this result for $y'(0)$ back into the formula for $y'(x)$.

$$y'(x) = - \int_0^1 u(t) dt + \int_0^x u(t) dt$$

Integrate both sides from 0 to x again.

$$\begin{aligned} \int_0^x y'(r) dr &= \int_0^x \left[- \int_0^1 u(t) dt + \int_0^r u(t) dt \right] dr \\ y(x) - y(0) &= -x \int_0^1 u(t) dt + \int_0^x \int_0^r u(t) dt dr \end{aligned}$$

Substitute $y(0) = 1$ and bring it to the right side.

$$y(x) = 1 - x \int_0^1 u(t) dt + \int_0^x \int_0^r u(t) dt dr$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned} v &= \int_0^r u(t) dt & dw &= dr \\ dv &= u(r) dr & w &= r \end{aligned}$$

and use the formula $\int v dw = vw - \int w dv$.

$$\begin{aligned}
 y(x) &= 1 - x \int_0^1 u(t) dt + r \int_0^r u(t) dt \Big|_0^x - \int_0^x ru(r) dr \\
 &= 1 - x \int_0^1 u(t) dt + x \int_0^x u(t) dt - \int_0^x ru(r) dr \\
 &= 1 - x \int_0^1 u(t) dt + x \int_0^x u(t) dt - \int_0^x tu(t) dt \\
 &= 1 - x \int_0^1 u(t) dt + \int_0^x (x-t)u(t) dt
 \end{aligned} \tag{2}$$

Now plug equations (1) and (2) into the original ODE.

$$y'' + 4y = x \quad \rightarrow \quad u(x) + 4 \left[1 - x \int_0^1 u(t) dt + \int_0^x (x-t)u(t) dt \right] = x$$

Expand the left side.

$$u(x) + 4 - 4x \int_0^1 u(t) dt + 4 \int_0^x (x-t)u(t) dt = x$$

Solve for $u(x)$.

$$\begin{aligned}
 u(x) &= -4 + x + 4x \int_0^1 u(t) dt - 4 \int_0^x (x-t)u(t) dt \\
 &= -4 + x + \int_0^1 4xu(t) dt - \int_0^x 4(x-t)u(t) dt \\
 &= -4 + x + \int_0^x 4xu(t) dt + \int_x^1 4xu(t) dt - \int_0^x 4(x-t)u(t) dt \\
 &= -4 + x + \int_0^x [4x - 4(x-t)]u(t) dt + \int_x^1 4xu(t) dt \\
 &= -4 + x + \int_0^x 4tu(t) dt + \int_x^1 4xu(t) dt
 \end{aligned}$$

Therefore, the equivalent Fredholm integral equation is

$$u(x) = -4 + x + \int_0^1 K(x,t)u(t) dt,$$

where

$$K(x,t) = \begin{cases} 4t & 0 \leq t \leq x \\ 4x & x \leq t \leq 1 \end{cases}.$$