Exercise 9

Convert each of the following Fredholm integral equation in 9–16 to an equivalent BVP:

$$u(x) = e^{2x} + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 3t(1-x), & \text{for } 0 \le t \le x \\ 3x(1-t), & \text{for } x \le t \le 1 \end{cases}$$

Solution

Substitute the given kernel $K(x,t)$ into the integral.

$$u(x) = e^{2x} + \int_0^x 3t(1-x)u(t)\,dt + \int_x^1 3x(1-t)u(t)\,dt \tag{1}$$

Differentiate both sides with respect to x.

$$u'(x) = 2e^{2x} + \frac{d}{dx}\int_0^x 3t(1-x)u(t)\,dt + \frac{d}{dx}\int_x^1 3x(1-t)u(t)\,dt$$

Apply the Leibniz rule to differentiate the integrals.

$$\begin{aligned} &= 2e^{2x} + \int_0^x \frac{\partial}{\partial x} 3t(1-x)u(t)\,dt + \cancel{3x(1-x)u(x)\cdot 1} - 3(0)(1-x)u(0)\cdot 0 \\ &\qquad + \int_x^1 \frac{\partial}{\partial x} 3x(1-t)u(t)\,dt + 3x(0)u(1)\cdot 0 - \cancel{3x(1-x)u(x)\cdot 1} \\ &= 2e^{2x} + \int_0^x (-3t)u(t)\,dt + \int_x^1 3(1-t)u(t)\,dt \\ &= 2e^{2x} - 3\int_0^x tu(t)\,dt - 3\int_1^x (1-t)u(t)\,dt \end{aligned}$$

Differentiate both sides with respect to x once more.

$$\begin{aligned} u''(x) &= 4e^{2x} - 3\frac{d}{dx}\int_0^x tu(t)\,dt - 3\frac{d}{dx}\int_1^x (1-t)u(t)\,dt \\ &= 4e^{2x} - 3xu(x) - 3(1-x)u(x) \\ &= 4e^{2x} - \cancel{3xu(x)} - 3u(x) + \cancel{3xu(x)} \end{aligned}$$

The boundary conditions are found by setting $x = 0$ and $x = 1$ in equation (1).

$$u(0) = e^0 + \int_0^0 3t(1)u(t)\,dt + \int_0^1 3(0)(1-t)u(t)\,dt = 1$$

$$u(1) = e^2 + \int_0^1 3t(0)u(t)\,dt + \int_1^1 3(1)(1-t)u(t)\,dt = e^2$$

Therefore, the equivalent BVP is

$$u'' + 3u = 4e^{2x},\ u(0) = 1,\ u(1) = e^2.$$

[TYPO: The answer at the back of the book reads $u'' + 3u = 4e^x,\ 0 < x < 1,\ u(0) = 1,\ u(1) = e^2$.]