Exercise 1

In Exercises 1–4, show that the given function $u(x)$ is a solution of the corresponding Fredholm integral equation:

$$u(x) = \cos x + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x u(t)\,dt,\ u(x) = \sin x + \cos x$$

Solution

Substitute the function in question on both sides of the integral equation.

$$\sin x + \cos x \stackrel{?}{=} \cos x + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x(\sin t + \cos t)\,dt$$

$\sin x$ can be brought in front of the integral since it doesn't depend on t.

$$\sin x + \cos x \stackrel{?}{=} \cos x + \frac{1}{2}\sin x\int_0^{\frac{\pi}{2}} (\sin t + \cos t)\,dt$$

Split up the integral into two.

$$\sin x + \cos x \stackrel{?}{=} \cos x + \frac{1}{2}\sin x\left(\int_0^{\frac{\pi}{2}} \sin t\,dt + \int_0^{\frac{\pi}{2}} \cos t\,dt\right)$$

Evaluate the integrals.

$$\begin{aligned}
\sin x + \cos x &\stackrel{?}{=} \cos x + \frac{1}{2}\sin x\left[(-\cos t)\Big|_0^{\pi/2} + (\sin t)\Big|_0^{\pi/2}\right] \\
&\stackrel{?}{=} \cos x + \frac{1}{2}\sin x\left[-\cos\frac{\pi}{2} - (-\cos 0) + \sin\frac{\pi}{2} - \sin 0\right] \\
&\stackrel{?}{=} \cos x + \frac{1}{2}\sin x(0 + 1 + 1 - 0) \\
&\stackrel{?}{=} \cos x + \frac{1}{2}\sin x(2) \\
&= \cos x + \sin x
\end{aligned}$$

Therefore,

$$u(x) = \cos x + \sin x$$

is a solution of the Fredholm integral equation.