Exercise 11

In Exercises 9–12, show that the given function $u(x)$ is a solution of the corresponding Fredholm integro-differential equation:

$$u''(x) = 1 - \sin x - \int_0^{\frac{\pi}{2}} tu(t)\,dt,\ u(0) = 0,\ u'(0) = 1,\ u(x) = \sin x$$

Solution

Substitute the function in question on both sides of the integro-differential equation.

$$\begin{aligned}
\frac{d^2}{dx^2}(\sin x) &\stackrel{?}{=} 1 - \sin x - \int_0^{\frac{\pi}{2}} t \sin t\,dt \\
-\sin x &\stackrel{?}{=} 1 - \sin x - \int_0^{\frac{\pi}{2}} t \sin t\,dt
\end{aligned}$$

Use integration by parts to solve the integral. Let

$$\begin{aligned}
v &= t & dw &= \sin t\,dt \\
dv &= dt & w &= (-\cos t)
\end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned}
-\sin x &\stackrel{?}{=} 1 - \sin x - \Bigg[\underbrace{-t\cos t\Big|_0^{\pi/2}}_{=0} - \int_0^{\pi/2} (-\cos t)\,dt\Bigg] \\
&\stackrel{?}{=} 1 - \sin x - \left(\int_0^{\pi/2} \cos t\,dt\right) \\
&\stackrel{?}{=} 1 - \sin x - (\sin t)\Big|_0^{\pi/2} \\
&\stackrel{?}{=} 1 - \sin x - \left(\sin\frac{\pi}{2} - 0\right) \\
&\stackrel{?}{=} 1 - \sin x - 1 \\
&= -\sin x
\end{aligned}$$

Therefore,

$$u(x) = \sin x$$

is a solution of the Fredholm integro-differential equation.