Exercise 14

In Exercises 13–16, show that the given function $u(x)$ is a solution of the corresponding Volterra integro-differential equation:

$$u''(x) = x\cos x - 2\sin x + \int_0^x tu(t)\,dt,\ u(0) = 0,\ u'(0) = 1,\ u(x) = \sin x$$

Solution

Substitute the function in question on both sides of the integro-differential equation.

$$\begin{aligned}
\frac{d^2}{dx^2}(\sin x) &\overset{?}{=} x\cos x - 2\sin x + \int_0^x t\sin t\,dt \\
-\sin x &\overset{?}{=} x\cos x - 2\sin x + \int_0^x t\sin t\,dt
\end{aligned}$$

To solve the integral, introduce a new variable s in the argument of sine.

$$\begin{aligned}
-\sin x &\overset{?}{=} x\cos x - 2\sin x + \int_0^x t\sin st\,dt\bigg|_{s=1} \\
&\overset{?}{=} x\cos x - 2\sin x + \int_0^x \frac{\partial}{\partial s}(-\cos st)\,dt\bigg|_{s=1} \\
&\overset{?}{=} x\cos x - 2\sin x - \frac{d}{ds}\int_0^x \cos st\,dt\bigg|_{s=1} \\
&\overset{?}{=} x\cos x - 2\sin x - \frac{d}{ds}\left[\frac{1}{s}(\sin st)\bigg|_0^x\right]\bigg|_{s=1} \\
&\overset{?}{=} x\cos x - 2\sin x - \frac{d}{ds}\left[\frac{1}{s}(\sin sx)\right]\bigg|_{s=1} \\
&\overset{?}{=} x\cos x - 2\sin x - \left[-\frac{1}{s^2}(\sin sx) + \frac{x}{s}\cos sx\right]\bigg|_{s=1} \\
&\overset{?}{=} x\cos x - 2\sin x - \Big(-\sin x + x\cos x\Big) \\
&\overset{?}{=} \cancel{x\cos x} - 2\sin x + \sin x - \cancel{x\cos x} \\
&= -\sin x
\end{aligned}$$

Therefore,

$$u(x) = \sin x$$

is a solution of the Volterra integro-differential equation.