Exercise 15

In Exercises 13–16, show that the given function $u(x)$ is a solution of the corresponding Volterra integro-differential equation:

$$u''(x) = 1 + \int_0^x (x-t)u(t)\,dt,\ u(0) = 1,\ u'(0) = 0,\ u(x) = \cosh x$$

Solution

Substitute the function in question on both sides of the integro-differential equation.

$$\begin{aligned}\frac{d^2}{dx^2}(\cosh x) &\stackrel{?}{=} 1 + \int_0^x (x-t)\cosh t\,dt \\ \cosh x &\stackrel{?}{=} 1 + \int_0^x (x-t)\cosh t\,dt\end{aligned}$$

Use integration by parts to solve the integral. Let

$$\begin{aligned}v &= x - t & dw &= \cosh t\,dt \\ dv &= -dt & w &= \sinh t\end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned}\cosh x &\stackrel{?}{=} 1 + \underbrace{(x-t)\sinh t\bigg|_0^x}_{=0} - \int_0^x \sinh t(-dt) \\ &\stackrel{?}{=} 1 + \int_0^x \sinh t\,dt \\ &\stackrel{?}{=} 1 + (\cosh t)\bigg|_0^x \\ &\stackrel{?}{=} 1 + (\cosh x - 1) \\ &= \cosh x\end{aligned}$$

Therefore,

$$u(x) = \cosh x$$

is a solution of the Volterra integro-differential equation.