Exercise 2

In Exercises 1–4, show that the given function $u(x)$ is a solution of the corresponding Fredholm integral equation:

$$u(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t}u(t)\,dt,\ u(x) = e^{2x}$$

Solution

Substitute the function in question on both sides of the integral equation.

$$\begin{aligned} e^{2x} &\stackrel{?}{=} e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t}e^{2t}\,dt \\ &\stackrel{?}{=} e^{2x}e^{\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x}e^{-\frac{5}{3}t}e^{2t}\,dt \end{aligned}$$

e^{2x} can be brought in front of the integral since it doesn't depend on t.

$$e^{2x} \stackrel{?}{=} e^{2x}e^{\frac{1}{3}} - \frac{1}{3}e^{2x}\int_0^1 e^{\frac{1}{3}t}\,dt$$

Divide both sides by e^{2x}.

$$1 \stackrel{?}{=} e^{\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{\frac{1}{3}t}\,dt$$

Evaluate the integral.

$$\begin{aligned} 1 &\stackrel{?}{=} e^{\frac{1}{3}} - \frac{\cancel{1}}{\cancel{3}}\cdot\frac{1}{\frac{\cancel{1}}{\cancel{3}}}e^{\frac{1}{3}t}\bigg|_0^1 \\ &\stackrel{?}{=} e^{\frac{1}{3}} - (e^{\frac{1}{3}} - e^0) \\ &\stackrel{?}{=} \cancel{e^{\frac{1}{3}}} - \cancel{e^{\frac{1}{3}}} + e^0 \\ &= 1 \end{aligned}$$

Therefore,

$$u(x) = e^{2x}$$

is a solution to the Fredholm integral equation.