Exercise 23

In Exercises 17–24, find the unknown if the solution of each equation is given:

$$\text{If } u(x) = 2 + 12x^2 \text{ is a solution of } u'(x) = f(x) + 20x - \int_0^x \int_0^1 (x-t)u(t)\,dt\,dt, \text{ find } f(x)$$

Solution

Substitute the solution into both sides of the equation.

$$\begin{aligned}
\frac{d}{dx}(2 + 12x^2) &= f(x) + 20x - \int_0^x \int_0^1 (x-t)(2+12t^2)\,dt\,dt \\
24x &= f(x) + 20x - \int_0^x \int_0^1 (2x + 12xt^2 - 2t - 12t^3)\,dt\,dt \\
&= f(x) + 20x - \int_0^x \left(\int_0^1 2x\,dt + \int_0^1 12xt^2\,dt - \int_0^1 2t\,dt - \int_0^1 12t^3\,dt \right) dt \\
&= f(x) + 20x - \int_0^x \left(2x + 12x \cdot \frac{t^3}{3}\bigg|_0^1 - 2 \cdot \frac{t^2}{2}\bigg|_0^1 - 12 \cdot \frac{t^4}{4}\bigg|_0^1 \right) dt \\
&= f(x) + 20x - \int_0^x (2x + 4x - 1 - 3)\,dt \\
&= f(x) + 20x - \int_0^x (6x - 4)\,dt
\end{aligned}$$

The meaning of the double differential is to put the result of the inner integral in terms of t and then integrate again.

$$\begin{aligned}
&= f(x) + 20x - \int_0^x (6t - 4)\,dt \\
&= f(x) + 20x - \left(\int_0^x 6t\,dt - \int_0^x 4\,dt \right) \\
&= f(x) + 20x - \left(6 \cdot \frac{t^2}{2}\bigg|_0^x - 4x \right) \\
&= f(x) + 20x - (3x^2 - 4x) \\
24x &= f(x) + 20x - 3x^2 + 4x
\end{aligned}$$

Subtract $24x$ from both sides.

$$0 = f(x) - 3x^2$$

Therefore,

$$f(x) = 3x^2.$$