Exercise 24

In Exercises 17–24, find the unknown if the solution of each equation is given:

$$\text{If } u(x) = 6x \text{ is a solution of } u(x) = f(x) + \int_0^x (1-t)u(t)\,dt - x\int_0^1 (x-t)u(t)\,dt\,dt, \text{ find } f(x)$$

[TYPO: This differential should not be here.]

Solution

Substitute the solution into both sides of the equation.

$$\begin{aligned}
6x &= f(x) + \int_0^x (1-t)6t\,dt - x\int_0^1 (x-t)6t\,dt \\
&= f(x) + \int_0^x (6t - 6t^2)\,dt - x\int_0^1 (6xt - 6t^2)\,dt \\
&= f(x) + \left(\int_0^x 6t\,dt - \int_0^x 6t^2\,dt\right) - x\left(\int_0^1 6xt\,dt - \int_0^1 6t^2\,dt\right) \\
&= f(x) + \left(6\cdot\frac{t^2}{2}\bigg|_0^x - 6\cdot\frac{t^3}{3}\bigg|_0^x\right) - x\left(6x\cdot\frac{t^2}{2}\bigg|_0^1 - 6\cdot\frac{t^3}{3}\bigg|_0^1\right) \\
&= f(x) + (3x^2 - 2x^3) - x(3x - 2) \\
&= f(x) + \cancel{3x^2} - 2x^3 - \cancel{3x^2} + 2x \\
6x &= f(x) - 2x^3 + 2x
\end{aligned}$$

Therefore,

$$f(x) = 4x + 2x^3.$$