Exercise 9

In Exercises 9–12, show that the given function $u(x)$ is a solution of the corresponding Fredholm integro-differential equation:

$$u'(x) = xe^x + e^x - x + \frac{1}{2}\int_0^1 xu(t)\,dt,\ u(0) = 0,\ u(x) = xe^x$$

[TYPO: The 1/2 should not be here.]

Solution

Substitute the function in question on both sides of the integro-differential equation.

$$\frac{d}{dx}(xe^x) \stackrel{?}{=} xe^x + e^x - x + \int_0^1 xte^t\,dt$$
$$e^x + xe^x \stackrel{?}{=} xe^x + e^x - x + x\int_0^1 te^t\,dt$$

Subtract $e^x + xe^x$ from both sides.

$$0 \stackrel{?}{=} -x + x\int_0^1 te^t\,dt$$

Use integration by parts.

$$\stackrel{?}{=} -x + x(te^t - e^t)\Big|_0^1$$
$$\stackrel{?}{=} -x + x(e^1 - e^1 - 0 + 1)$$
$$\stackrel{?}{=} -x + x$$
$$= 0$$

Therefore,

$$u(x) = xe^x$$

is a solution of the Fredholm integro-differential equation.