Exercise 16

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 1 - 2\int_0^x tu(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 1 - 2\int_0^x t\sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 - 2\int_0^x t[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1}_{u_0(x)} + \underbrace{\int_0^x (-2t)u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x (-2t)u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$.

$$\begin{aligned}
&u_0(x) = 1 \\
&u_1(x) = \int_0^x (-2t)u_0(t)\,dt = 2(-1)\int_0^x t(1)\,dt = 2(-1)\frac{x^2}{2} \\
&u_2(x) = \int_0^x (-2t)u_1(t)\,dt = 2^2(-1)^2\int_0^x t\left(\frac{t^2}{2}\right)\,dt = 2^2(-1)^2\frac{x^4}{4\cdot 2} \\
&u_3(x) = \int_0^x (-2t)u_2(t)\,dt = 2^3(-1)^3\int_0^x t\left(\frac{t^4}{4\cdot 2}\right)\,dt = 2^3(-1)^3\frac{x^6}{6\cdot 4\cdot 2} \\
&\qquad\vdots \\
&u_n(x) = \int_0^x (-2t)u_{n-1}(t)\,dt = 2^n(-1)^n\frac{x^{2n}}{(2n)!!} = 2^n(-1)^n\frac{x^{2n}}{2^n n!} = (-1)^n\frac{x^{2n}}{n!} = \frac{(-1)^n(x^2)^n}{n!} = \frac{(-x^2)^n}{n!}
\end{aligned}$$

Therefore,

$$u(x) = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = e^{-x^2}.$$