Exercise 18

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = -2 + 3x - x^2 - \int_0^x (x-t)u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = -2 + 3x - x^2 - \int_0^x (x-t) \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = -2 + 3x - x^2 - \int_0^x (x-t)[u_0(t) + u_1(t) + \cdots]\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{-2 + 3x - x^2}_{u_0(x)} + \underbrace{\int_0^x (-1)(x-t)u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x (-1)(x-t)u_1(t)\,dt}_{u_2(x)} + \cdots$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$. Note that the $(x-t)$ in the integrand essentially means that we integrate the function next to it twice.

$$u_0(x) = -2 + 3x - x^2$$

$$u_1(x) = \int_0^x (-1)(x-t)u_0(t)\,dt = (-1)\int_0^x (x-t)(-2+3t-t^2)\,dt = (-1)\left(-\frac{2x^2}{2\cdot 1} + \frac{3x^3}{3\cdot 2} - \frac{x^4}{4\cdot 3}\right)$$

$$u_2(x) = \int_0^x (-1)(x-t)u_1(t)\,dt = (-1)^2 \int_0^x (x-t)\left(-\frac{2t^2}{2\cdot 1} + \frac{3t^3}{3\cdot 2} - \frac{t^4}{4\cdot 3}\right)dt$$

$$= (-1)^2\left(-\frac{2x^4}{4\cdot 3\cdot 2\cdot 1} + \frac{3x^5}{5\cdot 4\cdot 3\cdot 2} - \frac{x^6}{6\cdot 5\cdot 4\cdot 3}\right)$$

$$u_3(x) = \int_0^x (-1)(x-t)u_2(t)\,dt = (-1)^3 \int_0^x (x-t)\left(-\frac{2t^4}{4\cdot 3\cdot 2\cdot 1} + \frac{3t^5}{5\cdot 4\cdot 3\cdot 2} - \frac{t^6}{6\cdot 5\cdot 4\cdot 3}\right)dt$$

$$= (-1)^3\left(-\frac{2x^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{3x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} - \frac{x^8}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}\right)$$

$$\vdots$$

$$u_n(x) = \int_0^x (-1)(x-t)u_{n-1}(t)\,dt = (-1)^n\left[-\frac{2x^{2n}}{(2n)!} + \frac{3x^{2n+1}}{(2n+1)!} - \frac{2x^{2n+2}}{(2n+2)!}\right]$$

$$= -2\frac{(-1)^n x^{2n}}{(2n)!} + 3\frac{(-1)^n x^{2n+1}}{(2n+1)!} - 2\frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

We have

$$\begin{aligned}
u(x) &= \sum_{n=0}^{\infty} u_n(x) \\
&= \sum_{n=0}^{\infty} \left[-2\frac{(-1)^n x^{2n}}{(2n)!} + 3\frac{(-1)^n x^{2n+1}}{(2n+1)!} - 2\frac{(-1)^n x^{2n+2}}{(2n+2)!}\right] \\
&= -2\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + 3\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} - 2\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \\
&= -2\cos x + 3\sin x + 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2(n+1)}}{[2(n+1)]!} \\
&= -2\cos x + 3\sin x + 2\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\
&= -2\cos x + 3\sin x + 2(\cos x - 1) \\
&= \cancel{-2\cos x} + 3\sin x + \cancel{2\cos x} - 2.
\end{aligned}$$

Therefore,

$$u(x) = 3\sin x - 2.$$