Exercise 21

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = -2 + 2x + x^2 + \int_0^x (x-t)u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= -2 + 2x + x^2 + \int_0^x (x-t)\sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= -2 + 2x + x^2 + \int_0^x (x-t)[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{-2 + 2x + x^2}_{u_0(x)} + \underbrace{\int_0^x (x-t)u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x (x-t)u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$. Note that the $(x-t)$ in the integrand essentially means that we integrate the function next to it twice.

$$\begin{aligned}
u_0(x) &= -2 + 2x + x^2 \\
u_1(x) &= \int_0^x (x-t)u_0(t)\,dt = \int_0^x (x-t)(-2+2t+t^2)\,dt = -\frac{2x^2}{2\cdot 1} + \frac{2x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3} \\
u_2(x) &= \int_0^x (x-t)u_1(t)\,dt = \int_0^x (x-t)\left(-\frac{2t^2}{2\cdot 1} + \frac{2t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3}\right)dt \\
&= -\frac{2x^4}{4\cdot 3\cdot 2\cdot 1} + \frac{2x^5}{5\cdot 4\cdot 3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 4\cdot 3} \\
u_3(x) &= \int_0^x (x-t)u_2(t)\,dt = \int_0^x (x-t)\left(-\frac{2t^4}{4\cdot 3\cdot 2\cdot 1} + \frac{2t^5}{5\cdot 4\cdot 3\cdot 2} + \frac{t^6}{6\cdot 5\cdot 4\cdot 3}\right)dt \\
&= -\frac{2x^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{2x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} + \frac{x^8}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3} \\
&\ \vdots \\
u_n(x) &= \int_0^x (x-t)u_{n-1}(t)\,dt = -\frac{2x^{2n}}{(2n)!} + \frac{2x^{2n+1}}{(2n+1)!} + \frac{2x^{2n+2}}{(2n+2)!}
\end{aligned}$$

We have

$$\begin{aligned}
u(x) &= \sum_{n=0}^{\infty}\left[-\frac{2x^{2n}}{(2n)!}+\frac{2x^{2n+1}}{(2n+1)!}+\frac{2x^{2n+2}}{(2n+2)!}\right] \\
&= -2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}+2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}+2\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+2)!} \\
&= -2\cosh x+2\sinh x+2\sum_{n=0}^{\infty}\frac{x^{2(n+1)}}{[2(n+1)]!} \\
&= -2\cosh x+2\sinh x+2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!} \\
&= -2\cosh x+2\sinh x+2(\cosh x-1) \\
&= -\cancel{2\cosh x}+2\sinh x+\cancel{2\cosh x}-2.
\end{aligned}$$

Therefore,

$$u(x) = 2(\sinh x - 1).$$