Exercise 26

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 1 + x^2 - \int_0^x (x - t + 1)^2 u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 1 + x^2 - \int_0^x (x - t + 1)^2 \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 + x^2 - \int_0^x (x - t + 1)^2 [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1 + x^2}_{u_0(x)} + \underbrace{\int_0^x (-1)(x - t + 1)^2 u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x (-1)(x - t + 1)^2 u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned}
u_0(x) &= 1 + x^2 \\
u_1(x) &= \int_0^x (-1)(x - t + 1)^2 u_0(t)\,dt = -x - x^2 - \frac{2}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 \\
u_2(x) &= \int_0^x (-1)(x - t + 1)^2 u_1(t)\,dt = \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \frac{2}{15}x^5 + \cdots \\
u_3(x) &= \int_0^x (-1)(x - t + 1)^2 u_2(t)\,dt = -\frac{1}{6}x^3 - \frac{1}{4}x^4 - \frac{1}{6}x^5 - \cdots \\
&\vdots
\end{aligned}$$

Adding the components together, we find that

$$\begin{aligned}
u(x) &= u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \\
&= 1 - x + \left(1 - 1 + \frac{1}{2}\right)x^2 + \left(-\frac{2}{3} + \frac{2}{3} - \frac{1}{6}\right)x^3 + \cdots \\
&= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots,
\end{aligned}$$

which is the beginning of the Taylor series for the exponential function. We venture the guess that $u(x) = e^{-x}$. Substitute this into the integral equation and see if both sides are equal. Note that $(x-t)^2/2$ in the integrand essentially means that we integrate the function next to it three times. Also, $(x-t)$ in the integrand means we integrate the function next to it twice.

$$\begin{aligned}
u(x) &= 1 + x^2 - \int_0^x (x-t+1)^2 u(t)\,dt \\
e^{-x} &\stackrel{?}{=} 1 + x^2 - \int_0^x (x-t+1)^2 e^{-t}\,dt \\
&\stackrel{?}{=} 1 + x^2 - \int_0^x [(x-t)^2 + 2(x-t) + 1]e^{-t}\,dt \\
&\stackrel{?}{=} 1 + x^2 - \left[2\int_0^x \frac{(x-t)^2}{2} e^{-t}\,dt + 2\int_0^x (x-t)e^{-t}\,dt + \int_0^x e^{-t}\,dt\right] \\
&\stackrel{?}{=} 1 + x^2 - \left[-2\int_0^x (x-t)(e^{-t}-1)\,dt - 2\int_0^x (e^{-t}-1)\,dt - (e^{-x}-1)\right] \\
&\stackrel{?}{=} 1 + x^2 - \left\{-2\int_0^x [-(e^{-t}-1)-t]\,dt - 2[-(e^{-x}-1)-x] - e^{-x} + 1\right\} \\
&\stackrel{?}{=} 1 + x^2 - \left[-2\int_0^x (-e^{-t}+1-t)\,dt + 2e^{-x} - 2 + 2x - e^{-x} + 1\right] \\
&\stackrel{?}{=} 1 + x^2 - \left\{-2\left[(e^{-x}-1) + x - \frac{x^2}{2}\right] + e^{-x} + 2x - 1\right\} \\
&\stackrel{?}{=} 1 + x^2 - \left(-2e^{-x} + 2 - \cancel{2x} + x^2 + e^{-x} + \cancel{2x} - 1\right) \\
&\stackrel{?}{=} 1 + x^2 - \left(x^2 - e^{-x} + 1\right) \\
&\stackrel{?}{=} \cancel{1} + \cancel{x^2} - \cancel{x^2} + e^{-x} - \cancel{1} \\
&= e^{-x}
\end{aligned}$$

Therefore,

$$u(x) = e^{-x}.$$