

Exercise 27

In Exercises 27–30, use the *Adomian decomposition method* to find the series solution

$$u(x) = 3 + \frac{1}{4} \int_0^x xt^2 u(t) dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= 3 + \frac{1}{4} \int_0^x xt^2 \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= 3 + \frac{1}{4} \int_0^x xt^2 [u_0(t) + u_1(t) + \cdots] dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{3}_{u_0(x)} + \underbrace{\frac{1}{4} \int_0^x xt^2 u_0(t) dt}_{u_1(x)} + \underbrace{\frac{1}{4} \int_0^x xt^2 u_1(t) dt}_{u_2(x)} + \cdots \end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned} u_0(x) &= 3 \\ u_1(x) &= \frac{1}{4} \int_0^x xt^2 u_0(t) dt = \frac{1}{4} \int_0^x xt^2 (3) dt = \frac{3}{4} x \cdot \frac{x^3}{3} = \frac{x^4}{4} \\ u_2(x) &= \frac{1}{4} \int_0^x xt^2 u_1(t) dt = \frac{1}{4} \int_0^x xt^2 \left(\frac{t^4}{4} \right) dt = \frac{1}{16} x \cdot \frac{x^7}{7} = \frac{x^8}{112} \\ u_3(x) &= \frac{1}{4} \int_0^x xt^2 u_2(t) dt = \frac{1}{4} \int_0^x xt^2 \left(\frac{t^8}{112} \right) dt = \frac{1}{448} x \cdot \frac{x^{11}}{11} = \frac{x^{12}}{4928} \\ &\vdots \end{aligned}$$

Adding the components together, therefore,

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \\ &= 3 + \frac{x^4}{4} + \frac{x^8}{112} + \frac{x^{12}}{4928} + \cdots \end{aligned}$$