

Exercise 30

In Exercises 27–30, use the *Adomian decomposition method* to find the series solution

$$u(x) = 1 + \frac{1}{2} \int_0^x x^2 u(t) dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= 1 + \frac{1}{2} \int_0^x x^2 \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 + \frac{1}{2} \int_0^x x^2 [u_0(t) + u_1(t) + \cdots] dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1}_{u_0(x)} + \underbrace{\frac{1}{2} \int_0^x x^2 u_0(t) dt}_{u_1(x)} + \underbrace{\frac{1}{2} \int_0^x x^2 u_1(t) dt}_{u_2(x)} + \cdots \end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned} u_0(x) &= 1 \\ u_1(x) &= \frac{1}{2} \int_0^x x^2 u_0(t) dt = \frac{1}{2} \int_0^x x^2 (1) dt = \frac{x^2}{2} \cdot x = \frac{1}{2} x^3 \\ u_2(x) &= \frac{1}{2} \int_0^x x^2 u_1(t) dt = \frac{1}{2} \int_0^x x^2 \left(\frac{1}{2} t^3 \right) dt = \frac{x^2}{4} \left(\frac{x^4}{4} \right) = \frac{1}{16} x^6 \\ u_3(x) &= \frac{1}{2} \int_0^x x^2 u_2(t) dt = \frac{1}{2} \int_0^x x^2 \left(\frac{1}{16} t^6 \right) dt = \frac{x^2}{32} \left(\frac{x^7}{7} \right) = \frac{1}{224} x^9 \\ &\vdots \end{aligned}$$

Adding the components together, therefore,

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) + u_2(x) + \cdots \\ &= 1 + \frac{1}{2} x^3 + \frac{1}{16} x^6 + \frac{1}{224} x^9 + \cdots \end{aligned}$$