Exercise 7

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 1 + x - \int_0^x (x-t)u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 1 + x - \int_0^x (x-t)\sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 + x - \int_0^x (x-t)[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1 + x}_{u_0(x)} + \underbrace{\int_0^x (-1)(x-t)u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x (-1)(x-t)u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$. Note that the $(x-t)$ in the integrand essentially means that we integrate the function next to it twice.

$$\begin{aligned}
u_0(x) &= 1 + x \\
u_1(x) &= \int_0^x (-1)(x-t)u_0(t)\,dt = (-1)\int_0^x (x-t)(1+t)\,dt = (-1)\left(\frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2}\right) \\
u_2(x) &= \int_0^x (-1)(x-t)u_1(t)\,dt = (-1)^2\int_0^x (x-t)\left(\frac{t^2}{2\cdot 1} + \frac{t^3}{3\cdot 2}\right)dt = (-1)^2\left(\frac{x^4}{4\cdot 3\cdot 2\cdot 1} + \frac{x^5}{5\cdot 4\cdot 3\cdot 2}\right) \\
u_3(x) &= \int_0^x (-1)(x-t)u_2(t)\,dt = (-1)^3\int_0^x (x-t)\left(\frac{t^4}{4\cdot 3\cdot 2\cdot 1} + \frac{t^5}{5\cdot 4\cdot 3\cdot 2}\right)dt \\
&= (-1)^3\left(\frac{x^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}\right) \\
&\vdots \\
u_n(x) &= \int_0^x (-1)(x-t)u_{n-1}(t)\,dt = (-1)^n\left[\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right] = \frac{(-1)^n x^{2n}}{(2n)!} + \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Therefore,

$$u(x) = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n}}{(2n)!} + \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos x + \sin x.$$