

Exercise 10

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = e^x - xe^x + \sin x + x \cos x + \int_0^x xu(t) dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= e^x - xe^x + \sin x + x \cos x + \int_0^x x \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= e^x + \sin x + x \cos x - xe^x + \int_0^x x[u_0(t) + u_1(t) + \cdots] dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{e^x + \sin x}_{u_0(x)} + \underbrace{x \cos x - xe^x + \int_0^x xu_0(t) dt}_{u_1(x)} + \underbrace{\int_0^x xu_1(t) dt}_{u_2(x)} + \cdots \end{aligned}$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned} u_0(x) &= e^x + \sin x \\ u_1(x) &= x \cos x - xe^x + \int_0^x xu_0(t) dt = x \cos x - xe^x + x(e^x - \cos x) = 0 \\ u_2(x) &= \int_0^x xu_1(t) dt = 0 \\ &\vdots \\ u_n(x) &= \int_0^x xu_{n-1}(t) dt = 0, \quad n > 2 \end{aligned}$$

Therefore,

$$u(x) = e^x + \sin x.$$