Exercise 11

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = 1 + x + x^2 + \frac{1}{2}x^3 + \cosh x + x\sinh x - \int_0^x xu(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = 1 + x + x^2 + \frac{1}{2}x^3 + \cosh x + x\sinh x - \int_0^x x\sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + x + \cosh x + x^2 + \frac{1}{2}x^3 + x\sinh x - \int_0^x x[u_0(t) + u_1(t) + \cdots]\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{1 + x + \cosh x}_{u_0(x)} + \underbrace{x^2 + \frac{1}{2}x^3 + x\sinh x - \int_0^x xu_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x x[-u_1(t)]\,dt}_{u_2(x)} + \cdots$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned}
u_0(x) &= 1 + x + \cosh x \\
u_1(x) &= x^2 + \frac{1}{2}x^3 + x\sinh x - \int_0^x xu_0(t)\,dt = x^2 + \frac{1}{2}x^3 + x\sinh x - x\left(x + \frac{x^2}{2} + \sinh x\right) = 0 \\
u_2(x) &= \int_0^x x[-u_1(t)]\,dt = 0 \\
&\vdots \\
u_n(x) &= \int_0^x x[-u_{n-1}(t)]\,dt = 0, \quad n > 2
\end{aligned}$$

Therefore,

$$u(x) = 1 + x + \cosh x.$$