Exercise 13

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = \sec x^2 - \left(1 - e^{\tan x}\right) x - x \int_0^x e^{\tan t} u(t)\, dt$$

[TYPO: The first term on the right side should be $\sec^2 x$ to get the answer at the back of the book.]

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = \sec^2 x - \left(1 - e^{\tan x}\right) x - x \int_0^x e^{\tan t} \sum_{n=0}^{\infty} u_n(t)\, dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \sec^2 x + \left(-1 + e^{\tan x}\right) x - x \int_0^x e^{\tan t} [u_0(t) + u_1(t) + \cdots]\, dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{\sec^2 x}_{u_0(x)} + \underbrace{\left(-1 + e^{\tan x}\right) x - x \int_0^x e^{\tan t} u_0(t)\, dt}_{u_1(x)} + \underbrace{x \int_0^x e^{\tan t} [-u_1(t)]\, dt}_{u_2(x)} + \cdots$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned}
u_0(x) &= \sec^2 x \\
u_1(x) &= \left(-1 + e^{\tan x}\right) x - x \int_0^x e^{\tan t} u_0(t)\, dt = \left(-1 + e^{\tan x}\right) x - x \left(e^{\tan x} - 1\right) = 0 \\
u_2(x) &= x \int_0^x e^{\tan t} [-u_1(t)]\, dt = 0 \\
&\vdots \\
u_n(x) &= x \int_0^x e^{\tan t} [-u_{n-1}(t)]\, dt = 0, \quad n > 2
\end{aligned}$$

Therefore,

$$u(x) = \sec^2 x.$$