Exercise 15

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = \sinh x + \frac{1}{10}(e - e^{\cosh x}) + \frac{1}{10}\int_0^x e^{\cosh t} u(t)\, dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = \sinh x + \frac{1}{10}(e - e^{\cosh x}) + \frac{1}{10}\int_0^x e^{\cosh t} \sum_{n=0}^{\infty} u_n(t)\, dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \sinh x + \frac{1}{10}(e - e^{\cosh x}) + \frac{1}{10}\int_0^x e^{\cosh t}[u_0(t) + u_1(t) + \cdots]\, dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{\sinh x}_{u_0(x)} + \underbrace{\frac{1}{10}(e - e^{\cosh x}) + \frac{1}{10}\int_0^x e^{\cosh t} u_0(t)\, dt}_{u_1(x)} + \underbrace{\frac{1}{10}\int_0^x e^{\cosh t} u_1(t)\, dt}_{u_2(x)} + \cdots$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned}
u_0(x) &= \sinh x \\
u_1(x) &= \frac{1}{10}(e - e^{\cosh x}) + \frac{1}{10}\int_0^x e^{\cosh t} u_0(t)\, dt = \frac{1}{10}(e - e^{\cosh x}) + \frac{1}{10}(e^{\cosh x} - e) = 0 \\
u_2(x) &= \frac{1}{10}\int_0^x e^{\cosh t} u_1(t)\, dt = 0 \\
&\vdots \\
u_n(x) &= \frac{1}{10}\int_0^x e^{\cosh t} u_{n-1}(t)\, dt = 0, \quad n > 2
\end{aligned}$$

Therefore,

$$u(x) = \sinh x.$$