

Exercise 2

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = \sinh x + \cosh x - 1 - \int_0^x u(t) dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= \sinh x + \cosh x - 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \dots &= \sinh x + \cosh x - 1 - \int_0^x [u_0(t) + u_1(t) + \dots] dt \\ u_0(x) + u_1(x) + u_2(x) + \dots &= \underbrace{\sinh x}_{u_0(x)} + \underbrace{\cosh x - 1 - \int_0^x u_0(t) dt}_{u_1(x)} + \underbrace{\int_0^x [-u_1(t)] dt}_{u_2(x)} + \dots \end{aligned}$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned} u_0(x) &= \sinh x \\ u_1(x) &= \cosh x - 1 - \int_0^x u_0(t) dt = \cosh x - 1 - (\cosh x - 1) = 0 \\ u_2(x) &= \int_0^x [-u_1(t)] dt = 0 \\ &\vdots \\ u_n(x) &= \int_0^x [-u_{n-1}(t)] dt = 0, \quad n > 2 \end{aligned}$$

Therefore,

$$u(x) = \sinh x.$$