Exercise 5

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = 2x - (1 - e^{-x^2}) + \int_0^x e^{-x^2+t^2} u(t)\, dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 2x - (1 - e^{-x^2}) + \int_0^x e^{-x^2+t^2} \sum_{n=0}^{\infty} u_n(t)\, dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 2x + (-1 + e^{-x^2}) + \int_0^x e^{-x^2+t^2} [u_0(t) + u_1(t) + \cdots]\, dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{2x}_{u_0(x)} + \underbrace{(-1 + e^{-x^2}) + \int_0^x e^{-x^2+t^2} u_0(t)\, dt}_{u_1(x)} + \underbrace{\int_0^x e^{-x^2+t^2} u_1(t)\, dt}_{u_2(x)} + \cdots
\end{aligned}$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned}
u_0(x) &= 2x \\
u_1(x) &= (-1 + e^{-x^2}) + \int_0^x e^{-x^2+t^2} u_0(t)\, dt = (-1 + e^{-x^2}) + (1 - e^{-x^2}) = 0 \\
u_2(x) &= \int_0^x e^{-x^2+t^2} u_1(t)\, dt = 0 \\
&\vdots \\
u_n(x) &= \int_0^x e^{-x^2+t^2} u_{n-1}(t)\, dt = 0, \quad n > 2
\end{aligned}$$

Therefore,

$$u(x) = 2x.$$