

Exercise 1

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = 6x + 3x^2 - \int_0^x u(t) dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= 6x + 3x^2 - \int_0^x \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \dots &= 6x + 3x^2 - \int_0^x [u_0(t) + u_1(t) + \dots] dt \\ u_0(x) + u_1(x) + u_2(x) + \dots &= \underbrace{6x + 3x^2}_{u_0(x)} + \underbrace{\int_0^x [-u_0(t)] dt}_{u_1(x)} + \underbrace{\int_0^x [-u_1(t)] dt}_{u_2(x)} + \dots \end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned} u_0(x) &= 6x + 3x^2 \\ u_1(x) &= \int_0^x [-u_0(t)] dt = - \int_0^x (6t + 3t^2) dt = -3x^2 - x^3 \\ &\vdots \end{aligned}$$

The noise terms $\pm 3x^2$ appear in both $u_0(x)$ and $u_1(x)$. Cancelling $3x^2$ from $u_0(x)$ leaves $6x$. Now we check to see whether $u(x) = 6x$ satisfies the integral equation.

$$\begin{aligned} 6x &\stackrel{?}{=} 6x + 3x^2 - \int_0^x 6t dt \\ 6x &\stackrel{?}{=} 6x + 3x^2 - 3x^2 \\ 6x &= 6x \end{aligned}$$

Therefore,

$$u(x) = 6x.$$