Exercise 4

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = x + x^2 - 2x^3 - x^4 + 12\int_0^x (x-t)u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = x + x^2 - 2x^3 - x^4 + 12\int_0^x (x-t)\sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = x + x^2 - 2x^3 - x^4 + 12\int_0^x (x-t)[u_0(t) + u_1(t) + \cdots]\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{x + x^2 - 2x^3 - x^4}_{u_0(x)} + \underbrace{12\int_0^x (x-t)u_0(t)\,dt}_{u_1(x)} + \underbrace{12\int_0^x (x-t)u_1(t)\,dt}_{u_2(x)} + \cdots$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. Note that having $(x-t)$ in the integrand means we integrate the function next to it twice.

$$\begin{aligned} u_0(x) &= x + x^2 - 2x^3 - x^4 \\ u_1(x) &= 12\int_0^x (x-t)u_0(t)\,dt = 12\left(\frac{x^3}{6} + \frac{x^4}{12} - \frac{1}{10}x^5 - \frac{1}{30}x^6\right) = 2x^3 + x^4 - \frac{6}{5}x^5 - \frac{2}{5}x^6 \\ &\vdots \end{aligned}$$

The noise terms, $\mp 2x^3$ and $\mp x^4$, appear in both $u_0(x)$ and $u_1(x)$. Cancelling $-2x^3$ and $-x^4$ from $u_0(x)$ leaves $x + x^2$. Now we check to see whether $u(x) = x + x^2$ satisfies the integral equation.

$$\begin{aligned} x + x^2 &\stackrel{?}{=} x + x^2 - 2x^3 - x^4 + 12\int_0^x (x-t)(t+t^2)\,dt \\ x + x^2 &\stackrel{?}{=} x + x^2 - 2x^3 - x^4 + 12\left(\frac{x^3}{6} + \frac{x^4}{12}\right) \\ x + x^2 &\stackrel{?}{=} x + x^2 - \cancel{2x^3} - \cancel{x^4} + \cancel{2x^3} + \cancel{x^4} \\ x + x^2 &= x + x^2 \end{aligned}$$

Therefore,

$$u(x) = x + x^2.$$