Exercise 7

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = \sinh x + x\sinh x - x^2\cosh x + \int_0^x xtu(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x) &= \sinh x + x\sinh x - x^2\cosh x + \int_0^x xt\sum_{n=0}^{\infty} u_n(t)\,dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \sinh x + x\sinh x - x^2\cosh x + \int_0^x xt[u_0(t) + u_1(t) + \cdots]\,dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{\sinh x + x\sinh x - x^2\cosh x}_{u_0(x)} + \underbrace{x\int_0^x tu_0(t)\,dt}_{u_1(x)} + \underbrace{x\int_0^x tu_1(t)\,dt}_{u_2(x)} + \cdots\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned}u_0(x) &= \sinh x + x\sinh x - x^2\cosh x \\ u_1(x) &= x\int_0^x tu_0(t)\,dt = x\int_0^x t(\sinh t + t\sinh t - t^2\cosh t)\,dt \\ &= -8x + 8x\cosh x + x^2\cosh x + 4x^3\cosh x - x\sinh x - 8x^2\sinh x - x^4\sinh x \\ &\vdots\end{aligned}$$

The noise terms, $\pm x\sinh x$ and $\mp x^2\cosh x$, appear in both $u_0(x)$ and $u_1(x)$. Cancelling $x\sinh x$ and $-x^2\cosh x$ from $u_0(x)$ leaves $\sinh x$. Now we check to see whether $u(x) = \sinh x$ satisfies the integral equation.

$$\begin{aligned}\sinh x &\stackrel{?}{=} \sinh x + x\sinh x - x^2\cosh x + \int_0^x xt\sinh t\,dt \\ \sinh x &\stackrel{?}{=} \sinh x + \cancel{x\sinh x} - \cancel{x^2\cosh x} + x(\cancel{x\cosh x} - \cancel{\sinh x}) \\ \sinh x &= \sinh x\end{aligned}$$

Therefore,

$$u(x) = \sinh x.$$