Exercise 9

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = \sec^2 x - \tan x + \int_0^x u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= \sec^2 x - \tan x + \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \sec^2 x - \tan x + \int_0^x [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{\sec^2 x - \tan x}_{u_0(x)} + \underbrace{\int_0^x u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned}
u_0(x) &= \sec^2 x - \tan x \\
u_1(x) &= \int_0^x u_0(t)\,dt = \int_0^x (\sec^2 t - \tan t)\,dt = \int_0^x \left(\frac{d}{dt}\tan t - \tan t\right) dt = \tan x - \ln|\sec x| \\
&\ \ \vdots
\end{aligned}$$

The noise terms $\mp \tan x$ appear in both $u_0(x)$ and $u_1(x)$. Cancelling $-\tan x$ from $u_0(x)$ leaves $\sec^2 x$. Now we check to see whether $u(x) = \sec^2 x$ satisfies the integral equation.

$$\begin{aligned}
\sec^2 x &\stackrel{?}{=} \sec^2 x - \tan x + \int_0^x \sec^2 t\,dt \\
\sec^2 x &\stackrel{?}{=} \sec^2 x - \tan x + \tan x \\
\sec^2 x &= \sec^2 x
\end{aligned}$$

Therefore,

$$u(x) = \sec^2 x.$$