Exercise 1

Use the *successive approximations method* to solve the following Volterra integral equations:

$$u(x) = x + \int_0^x u(t)\,dt$$

Solution

The successive approximations method, also known as the Picard iteration method, will be used to solve the integral equation. Consider the iteration scheme,

$$u_{n+1}(x) = x + \int_0^x u_n(t)\,dt, \quad n \geq 0,$$

choosing $u_0(x) = 0$. Then

$$\begin{aligned}
u_1(x) &= x + \int_0^x u_0(t)\,dt = x \\
u_2(x) &= x + \int_0^x u_1(t)\,dt = x + \frac{1}{2}x^2 \\
u_3(x) &= x + \int_0^x u_2(t)\,dt = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\
u_4(x) &= x + \int_0^x u_3(t)\,dt = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 \\
&\vdots,
\end{aligned}$$

and the general formula for $u_{n+1}(x)$ is

$$u_{n+1}(x) = \sum_{k=1}^{n+1} \frac{x^k}{k!}.$$

Take the limit as $n \to \infty$ to determine $u(x)$.

$$\begin{aligned}
\lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \sum_{k=1}^{n+1} \frac{x^k}{k!} \\
&= \sum_{k=1}^{\infty} \frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 \\
&= e^x - 1
\end{aligned}$$

Therefore, $u(x) = e^x - 1$.