Exercise 4

Use the *successive approximations method* to solve the following Volterra integral equations:

$$u(x) = 1 + 2x + 4\int_0^x (x-t)u(t)\,dt$$

Solution

The successive approximations method, also known as the method of Picard iteration, will be used to solve the integral equation. Consider the iteration scheme,

$$u_{n+1}(x) = 1 + 2x + 4\int_0^x (x-t)u_n(t)\,dt, \quad n \geq 0,$$

choosing $u_0(x) = 1$. Then

$$\begin{aligned}
u_1(x) &= 1 + 2x + 4\int_0^x (x-t)u_0(t)\,dt = 1 + 2x + 2x^2 \\
u_2(x) &= 1 + 2x + 4\int_0^x (x-t)u_1(t)\,dt = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 \\
u_3(x) &= 1 + 2x + 4\int_0^x (x-t)u_2(t)\,dt = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 \\
u_4(x) &= 1 + 2x + 4\int_0^x (x-t)u_3(t)\,dt = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{8}{315}x^7 + \frac{2}{315}x^8 \\
&\ \ \vdots,
\end{aligned}$$

and the general formula for $u_{n+1}(x)$ is

$$u_{n+1}(x) = \sum_{k=0}^{2n+2} \frac{(2x)^k}{k!}.$$

Take the limit as $n \to \infty$ to determine $u(x)$.

$$\begin{aligned}
\lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \sum_{k=0}^{2n+2} \frac{(2x)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} \\
&= e^{2x}
\end{aligned}$$

Therefore, $u(x) = e^{2x}$.