Exercise 6

Use the *successive approximations method* to solve the following Volterra integral equations:

$$u(x) = 1 + x^2 - \int_0^x (x - t + 1)^2 u(t)\, dt$$

Solution

The successive approximations method, also known as the method of Picard iteration, will be used to solve the integral equation. Consider the iteration scheme,

$$\begin{aligned} u_{n+1}(x) &= 1 + x^2 - \int_0^x (x - t + 1)^2 u_n(t)\, dt, \quad n \geq 0 \\ &= 1 + x^2 - \int_0^x [(x-t)^2 + 2(x-t) + 1] u_n(t)\, dt \\ &= 1 + x^2 - 2\int_0^x \frac{(x-t)^2}{2} u_n(t)\, dt - 2\int_0^x (x-t) u_n(t)\, dt - \int_0^x u_n(t)\, dt \\ &= 1 + x^2 - 2\int_0^x \int_0^r \int_0^s u_n(t)\, dt\, ds\, dr - 2\int_0^x \int_0^r u_n(t)\, dt\, dr - \int_0^x u_n(t)\, dt, \end{aligned}$$

choosing $u_0(x) = 1$. Then

$$\begin{aligned} u_1(x) &= 1 + x^2 - \int_0^x (x-t+1)^2 u_0(t)\, dt = 1 - x - \frac{1}{3}x^3 \\ u_2(x) &= 1 + x^2 - \int_0^x (x-t+1)^2 u_1(t)\, dt = 1 - x + \frac{1}{2}x^2 + \frac{1}{6}x^4 + \frac{1}{30}x^5 + \frac{1}{180}x^6 \\ u_3(x) &= 1 + x^2 - \int_0^x (x-t+1)^2 u_2(t)\, dt = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{20}x^5 - \frac{1}{60}x^6 - \frac{1}{252}x^7 \\ &\qquad - \frac{1}{2520}x^8 - \frac{1}{45\,360}x^9 \\ u_4(x) &= 1 + x^2 - \int_0^x (x-t+1)^2 u_3(t)\, dt = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{90}x^6 + \frac{1}{210}x^7 \\ &\qquad + \frac{1}{4536}x^9 + \frac{1}{45\,360}x^{10} + \frac{1}{831\,600}x^{11} + \frac{1}{29\,937\,600}x^{12} \\ &\ \ \vdots, \end{aligned}$$

and the general formula for $u_{n+1}(x)$ is

$$u_{n+1}(x) = \sum_{k=0}^{n+1} \frac{(-1)^k}{k!} x^k \pm \text{terms that vanish as } n \to \infty.$$

Take the limit as $n \to \infty$ to determine $u(x)$.

$$\begin{aligned} \lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \left(\sum_{k=0}^{n+1} \frac{(-1)^k}{k!} x^k \pm \text{terms that vanish as } n \to \infty \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \\ &= e^{-x} \end{aligned}$$

Therefore, $u(x) = e^{-x}$.