Exercise 7

Use the *successive approximations method* to solve the following Volterra integral equations:

$$u(x) = \frac{1}{2}x^2 - \int_0^x (x-t)u(t)\,dt$$

Solution

The successive approximations method, also known as the method of Picard iteration, will be used to solve the integral equation. Consider the iteration scheme,

$$u_{n+1}(x) = \frac{1}{2}x^2 - \int_0^x (x-t)u_n(t)\,dt, \quad n \geq 0,$$

choosing $u_0(x) = 0$. Then

$$\begin{aligned}
u_1(x) &= \frac{1}{2}x^2 - \int_0^x (x-t)u_0(t)\,dt = \frac{1}{2}x^2 \\
u_2(x) &= \frac{1}{2}x^2 - \int_0^x (x-t)u_1(t)\,dt = \frac{1}{2}x^2 - \frac{1}{24}x^4 \\
u_3(x) &= \frac{1}{2}x^2 - \int_0^x (x-t)u_2(t)\,dt = \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 \\
u_4(x) &= \frac{1}{2}x^2 - \int_0^x (x-t)u_3(t)\,dt = \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \frac{1}{40\,320}x^8 \\
&\ \ \vdots,
\end{aligned}$$

and the general formula for $u_{n+1}(x)$ is

$$u_{n+1}(x) = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{(2k)!}x^{2k}.$$

Take the limit as $n \to \infty$ to determine $u(x)$.

$$\begin{aligned}
\lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{(2k)!}x^{2k} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!}x^{2k} \\
&= -\sum_{k=1}^{\infty} \frac{(-1)^{k}}{(2k)!}x^{2k} \\
&= 1 - \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!}x^{2k} \\
&= 1 - \cos x
\end{aligned}$$

Therefore, $u(x) = 1 - \cos x$.