Exercise 14

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\begin{aligned}
\mathcal{L}\{u(x)\} &= \mathcal{L}\left\{2\cosh x - 2 + \int_0^x (x-t)u(t)\,dt\right\} \\
U(s) &= 2\mathcal{L}\{\cosh x\} - 2\mathcal{L}\{1\} + \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\} \\
&= 2\mathcal{L}\{\cosh x\} - 2\mathcal{L}\{1\} + \mathcal{L}\{x\}U(s) \\
&= 2\left(\frac{s}{s^2-1}\right) - 2\left(\frac{1}{s}\right) + \left(\frac{1}{s^2}\right)U(s)
\end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned}
\left(1 - \frac{1}{s^2}\right)U(s) &= \frac{2s}{s^2-1} - \frac{2}{s} \\
(s^2-1)U(s) &= \frac{2s^3}{s^2-1} - 2s \\
&= \frac{2s}{s^2-1} \\
U(s) &= \frac{2s}{(s^2-1)^2} \\
&= \frac{2s}{(s+1)^2(s-1)^2} = \frac{-\frac{1}{2}}{(s+1)^2} + \frac{\frac{1}{2}}{(s-1)^2}
\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}
u(x) &= \mathcal{L}^{-1}\{U(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{(s-1)^2}\right\} \\
&= -\frac{1}{2}xe^{-x} + \frac{1}{2}xe^{x} = x\left(\frac{e^x - e^{-x}}{2}\right) = x\sinh x
\end{aligned}$$