

Exercise 15

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = 2 - 2 \cos x - \int_0^x (x-t)u(t) dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^{\infty} e^{-sx} f(x) dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L} \left\{ \int_0^x f(x-t)g(t) dt \right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\begin{aligned} \mathcal{L}\{u(x)\} &= \mathcal{L} \left\{ 2 - 2 \cos x - \int_0^x (x-t)u(t) dt \right\} \\ U(s) &= 2\mathcal{L}\{1\} - 2\mathcal{L}\{\cos x\} - \mathcal{L} \left\{ \int_0^x (x-t)u(t) dt \right\} \\ &= 2\mathcal{L}\{1\} - 2\mathcal{L}\{\cos x\} - \mathcal{L}\{x\}U(s) \\ &= 2 \left(\frac{1}{s} \right) - 2 \left(\frac{s}{s^2+1} \right) - \left(\frac{1}{s^2} \right) U(s) \end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned} \left(1 + \frac{1}{s^2} \right) U(s) &= \frac{2}{s} - \frac{2s}{s^2+1} \\ (s^2+1)U(s) &= 2s - \frac{2s^3}{s^2+1} \\ &= \frac{2s}{s^2+1} \\ U(s) &= \frac{2s}{(s^2+1)^2} \\ &= -\frac{d}{ds} \left(\frac{1}{s^2+1} \right) \end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned} u(x) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \mathcal{L}^{-1} \left\{ -\frac{d}{ds} \left(\frac{1}{s^2+1} \right) \right\} \\ &= x \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} \\ &= x \sin x \end{aligned}$$