

Exercise 2

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = 1 - x - \int_0^x (x-t)u(t) dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^{\infty} e^{-sx} f(x) dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t) dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\begin{aligned}\mathcal{L}\{u(x)\} &= \mathcal{L}\left\{1 - x - \int_0^x (x-t)u(t) dt\right\} \\ U(s) &= \mathcal{L}\{1\} - \mathcal{L}\{x\} - \mathcal{L}\left\{\int_0^x (x-t)u(t) dt\right\} \\ &= \mathcal{L}\{1\} - \mathcal{L}\{x\} - \mathcal{L}\{x\}U(s) \\ &= \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^2}U(s)\end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned}\left(1 + \frac{1}{s^2}\right)U(s) &= \frac{1}{s} - \frac{1}{s^2} \\ U(s) &= \frac{\frac{1}{s} - \frac{1}{s^2}}{1 + \frac{1}{s^2}} \\ &= \frac{s-1}{s^2+1} \\ &= \frac{s}{s^2+1} - \frac{1}{s^2+1}\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}u(x) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \cos x - \sin x\end{aligned}$$