# Exercise 5

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = x - 1 + \int_0^x (x-t)u(t)\,dt$$

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**Solution**

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\begin{aligned}\mathcal{L}\{u(x)\} &= \mathcal{L}\left\{x - 1 + \int_0^x (x-t)u(t)\,dt\right\}\\ U(s) &= \mathcal{L}\{x\} - \mathcal{L}\{1\} + \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\}\\ &= \mathcal{L}\{x\} - \mathcal{L}\{1\} + \mathcal{L}\{x\}U(s)\\ &= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s^2}U(s)\end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned}\left(1 - \frac{1}{s^2}\right)U(s) &= \frac{1}{s^2} - \frac{1}{s}\\ U(s) &= \frac{\frac{1}{s^2} - \frac{1}{s}}{1 - \frac{1}{s^2}}\\ &= \frac{1-s}{s^2-1}\\ &= \frac{1}{s^2-1} - \frac{s}{s^2-1}\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}u(x) &= \mathcal{L}^{-1}\{U(s)\}\\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}\\ &= \sinh x - \cosh x\\ &= \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2}\\ &= -e^{-x}\end{aligned}$$