Exercise 1

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 1 - \int_0^x u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this into the integral equation.

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots &= 1 - \int_0^x (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)\,dt \\ &= 1 - a_0 x - \frac{a_1}{2} x^2 - \frac{a_2}{3} x^3 - \frac{a_3}{4} x^4 - \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned} &a_0 = 1 \\ &a_1 = -a_0 && \rightarrow && a_1 = -1 \\ &a_2 = -\frac{a_1}{2} && \rightarrow && a_2 = \frac{1}{2} \\ &a_3 = -\frac{a_2}{3} && \rightarrow && a_3 = -\frac{1}{6} \\ &a_4 = -\frac{a_3}{4} && \rightarrow && a_4 = \frac{1}{24} \\ &\quad\vdots && && \quad\vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots \\ &= e^{-x}. \end{aligned}$$