Exercise 11

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this and the Taylor series expansion of $\cosh x$,

$$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots ,$$

into the integral equation.

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + +a_9 x^9 + \cdots \\ = 2\left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots\right) - 2 \\ + \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + \cdots)\,dt \end{aligned}$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + +a_9 x^9 + \cdots \\ = x^2 + \frac{1}{12}x^4 + \frac{1}{360}x^6 + \frac{1}{20\,160}x^8 + \cdots \\ + \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \frac{a_6}{56}x^8 + \frac{a_7}{72}x^9 + \cdots \end{aligned}$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + +a_9 x^9 + \cdots \\ = \left(1 + \frac{a_0}{2}\right)x^2 + \frac{a_1}{6}x^3 + \left(\frac{1}{12} + \frac{a_2}{12}\right)x^4 + \frac{a_3}{20}x^5 \\ + \left(\frac{1}{360} + \frac{a_4}{30}\right)x^6 + \frac{a_5}{42}x^7 + \left(\frac{1}{20\,160} + \frac{a_6}{56}\right)x^8 + \frac{a_7}{72}x^9 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= 0 \\
a_2 &= 1 + \frac{a_0}{2} && \rightarrow && a_2 = 1 \\
a_3 &= \frac{a_1}{6} && \rightarrow && a_3 = 0 \\
a_4 &= \frac{1}{12} + \frac{a_2}{12} && \rightarrow && a_4 = \frac{1}{6} \\
a_5 &= \frac{a_3}{20} && \rightarrow && a_5 = 0 \\
a_6 &= \frac{1}{360} + \frac{a_4}{30} && \rightarrow && a_6 = \frac{1}{120} \\
a_7 &= \frac{a_5}{42} && \rightarrow && a_7 = 0 \\
a_8 &= \frac{1}{20\,160} + \frac{a_6}{56} && \rightarrow && a_8 = \frac{1}{5040} \\
a_9 &= \frac{a_7}{72} && \rightarrow && a_9 = 0 \\
&\vdots && && \vdots
\end{aligned}$$

So then

$$\begin{aligned}
u(x) &= x^2 + \frac{1}{6}x^4 + \frac{1}{120}x^6 + \frac{1}{5040}x^8 + \cdots \\
&= x\left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots\right) \\
&= x \sinh x.
\end{aligned}$$