

## Exercise 12

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 1 - x - \int_0^x (x-t)u(t) dt$$

### Solution

We seek a series solution for  $u$ :

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

Substitute this into the integral equation.

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \dots \\ = 1 - x - \int_0^x (x-t)(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + \dots) dt \end{aligned}$$

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \dots \\ = 1 - x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 - \frac{a_2}{12}x^4 - \frac{a_3}{20}x^5 - \frac{a_4}{30}x^6 - \frac{a_5}{42}x^7 - \dots \end{aligned}$$

Match the coefficients of the respective powers of  $x$  to determine  $a_i$ .

$$\begin{aligned} a_0 &= 1 \\ a_1 &= -1 \\ a_2 &= -\frac{a_0}{2} & \rightarrow & a_2 = -\frac{1}{2} \\ a_3 &= -\frac{a_1}{6} & \rightarrow & a_3 = \frac{1}{6} \\ a_4 &= -\frac{a_2}{12} & \rightarrow & a_4 = \frac{1}{24} \\ a_5 &= -\frac{a_3}{20} & \rightarrow & a_5 = -\frac{1}{120} \\ a_6 &= -\frac{a_4}{30} & \rightarrow & a_6 = -\frac{1}{720} \\ a_7 &= -\frac{a_5}{42} & \rightarrow & a_7 = \frac{1}{5040} \\ &\vdots & & \vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= 1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \dots \\ &= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots\right) - \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots\right) \\ &= \cos x - \sin x. \end{aligned}$$