Exercise 15

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = \sec x + \tan x - \int_0^x \sec t\, u(t)\, dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this and the Taylor series expansions of $\sec x$,

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots ,$$

and $\tan x$,

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots ,$$

into the integral equation.

$$\begin{aligned}
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \cdots \\
= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots \\
x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots \\
- \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{5}{24}t^4 + \frac{61}{720}t^6 + \cdots\right)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)\, dt
\end{aligned}$$

$$\begin{aligned}
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \cdots \\
= 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{2}{15}x^5 + \frac{61}{720}x^6 + \frac{17}{315}x^7 + \frac{277}{8064}x^8 + \frac{62}{2835}x^9 + \cdots \\
- \int_0^x (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 + \cdots)\, dt \\
- \frac{1}{2}\int_0^x t^2 (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots)\, dt \\
- \frac{5}{24}\int_0^x t^4 (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots)\, dt \\
- \frac{61}{720}\int_0^x t^6 (a_0 + a_1 t + a_2 t^2 + \cdots)\, dt - \frac{277}{8064}\int_0^x t^8 (a_0 + \cdots)\, dt - \cdots
\end{aligned}$$

$$\begin{aligned}
&a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{2}{15}x^5 + \frac{61}{720}x^6 + \frac{17}{315}x^7 + \frac{277}{8064}x^8 + \frac{62}{2835}x^9 + \cdots \\
&\quad - \left(a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \frac{a_6}{7}x^7 + \frac{a_7}{8}x^8 + \frac{a_8}{9}x^9 + \cdots\right) \\
&\quad - \frac{1}{2}\left(\frac{a_0}{3}x^3 + \frac{a_1}{4}x^4 + \frac{a_2}{5}x^5 + \frac{a_3}{6}x^6 + \frac{a_4}{7}x^7 + \frac{a_5}{8}x^8 + \frac{a_6}{9}x^9 + \cdots\right) \\
&\quad - \frac{5}{24}\left(\frac{a_0}{5}x^5 + \frac{a_1}{6}x^6 + \frac{a_2}{7}x^7 + \frac{a_3}{8}x^8 + \frac{a_4}{9}x^9 + \cdots\right) \\
&\quad - \frac{61}{720}\left(\frac{a_0}{7}x^7 + \frac{a_1}{8}x^8 + \frac{a_2}{9}x^9 + \cdots\right) - \frac{277}{8064}\left(\frac{a_0}{9}x^9 + \cdots\right) - \cdots
\end{aligned}$$

$$\begin{aligned}
&a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots \\
&= 1 + (1 - a_0)x + \left(\frac{1}{2} - \frac{a_1}{2}\right)x^2 + \left(\frac{1}{3} - \frac{a_2}{3} - \frac{a_0}{6}\right)x^3 \\
&\quad + \left(\frac{5}{24} - \frac{a_3}{4} - \frac{a_1}{8}\right)x^4 + \left(\frac{2}{15} - \frac{a_4}{5} - \frac{a_2}{10} - \frac{a_0}{24}\right)x^5 \\
&\quad + \left(\frac{61}{720} - \frac{a_5}{6} - \frac{a_3}{12} - \frac{5a_1}{144}\right)x^6 \\
&\quad + \left(\frac{17}{315} - \frac{a_6}{7} - \frac{a_4}{14} - \frac{5a_2}{168} - \frac{61a_0}{5040}\right)x^7 \\
&\quad + \left(\frac{277}{8064} - \frac{a_7}{8} - \frac{a_5}{16} - \frac{5a_3}{192} - \frac{61a_1}{5760}\right)x^8 \\
&\quad + \left(\frac{62}{2835} - \frac{a_8}{9} - \frac{a_6}{18} - \frac{5a_4}{216} - \frac{61a_2}{6480} - \frac{277a_0}{72\,576}\right)x^9 + \cdots
\end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 - a_0 & &\rightarrow \quad a_1 = 0 \\
a_2 &= \frac{1}{2} - \frac{a_1}{2} & &\rightarrow \quad a_2 = \frac{1}{2} \\
a_3 &= \frac{1}{3} - \frac{a_2}{3} - \frac{a_0}{6} & &\rightarrow \quad a_3 = 0 \\
a_4 &= \frac{5}{24} - \frac{a_3}{4} - \frac{a_1}{8} & &\rightarrow \quad a_4 = \frac{5}{24} \\
a_5 &= \frac{2}{15} - \frac{a_4}{5} - \frac{a_2}{10} - \frac{a_0}{24} & &\rightarrow \quad a_5 = 0 \\
a_6 &= \frac{61}{720} - \frac{a_5}{6} - \frac{a_3}{12} - \frac{5a_1}{144} & &\rightarrow \quad a_6 = \frac{61}{720} \\
a_7 &= \frac{17}{315} - \frac{a_6}{7} - \frac{a_4}{14} - \frac{5a_2}{168} - \frac{61a_0}{5040} & &\rightarrow \quad a_7 = 0 \\
a_8 &= \frac{277}{8064} - \frac{a_7}{8} - \frac{a_5}{16} - \frac{5a_3}{192} - \frac{61a_1}{5760} & &\rightarrow \quad a_8 = \frac{277}{8064} \\
a_9 &= \frac{62}{2835} - \frac{a_8}{9} - \frac{a_6}{18} - \frac{5a_4}{216} - \frac{61a_2}{6480} - \frac{277a_0}{72\,576} & &\rightarrow \quad a_9 = 0 \\
&\vdots & &\qquad\quad \vdots
\end{aligned}$$

So then

$$\begin{aligned} u(x) &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots \\ &= \sec x. \end{aligned}$$