Exercise 5

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 1 + xe^x - \int_0^x tu(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots .$$

Substitute this and the Taylor series expansion of e^x,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \cdots ,$$

into the integral equation.

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + x\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \cdots\right) \\ - \int_0^x t(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + \cdots)\,dt \end{aligned}$$

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6 + \frac{1}{720}x^7 + \cdots \\ - \frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 - \frac{a_2}{4}x^4 - \frac{a_3}{5}x^5 - \frac{a_4}{6}x^6 - \frac{a_5}{7}x^7 - \cdots \end{aligned}$$

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + x + \left(1 - \frac{a_0}{2}\right)x^2 + \left(\frac{1}{2} - \frac{a_1}{3}\right)x^3 + \left(\frac{1}{6} - \frac{a_2}{4}\right)x^4 \\ + \left(\frac{1}{24} - \frac{a_3}{5}\right)x^5 + \left(\frac{1}{120} - \frac{a_4}{6}\right)x^6 + \left(\frac{1}{720} - \frac{a_5}{7}\right)x^7 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 \\
a_2 &= 1 - \frac{a_0}{2} & \quad \rightarrow \quad & a_2 = \frac{1}{2} \\
a_3 &= \frac{1}{2} - \frac{a_1}{3} & \quad \rightarrow \quad & a_3 = \frac{1}{6} \\
a_4 &= \frac{1}{6} - \frac{a_2}{4} & \quad \rightarrow \quad & a_4 = \frac{1}{24} \\
a_5 &= \frac{1}{24} - \frac{a_3}{5} & \quad \rightarrow \quad & a_5 = \frac{1}{120} \\
a_6 &= \frac{1}{120} - \frac{a_4}{6} & \quad \rightarrow \quad & a_6 = \frac{1}{720} \\
a_7 &= \frac{1}{720} - \frac{a_5}{7} & \quad \rightarrow \quad & a_7 = \frac{1}{5040} \\
&\vdots & & \vdots
\end{aligned}
$$

So then

$$
\begin{aligned}
u(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \cdots \\
&= e^x.
\end{aligned}
$$