Exercise 6

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 1 + 2x + 4\int_0^x (x-t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Substitute this into the integral equation.

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + 2x + 4\int_0^x (x-t)(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + \cdots)\,dt \end{aligned}$$

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + 2x + 4\left(\frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \cdots\right) \end{aligned}$$

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + 2x + 2a_0x^2 + \frac{2a_1}{3}x^3 + \frac{a_2}{3}x^4 + \frac{a_3}{5}x^5 + \frac{2a_4}{15}x^6 + \frac{2a_5}{21}x^7 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 2 \\
a_2 &= 2a_0 & \rightarrow \quad a_2 &= 2 \\
a_3 &= \frac{2a_1}{3} & \rightarrow \quad a_3 &= \frac{4}{3} \\
a_4 &= \frac{a_2}{3} & \rightarrow \quad a_4 &= \frac{2}{3} \\
a_5 &= \frac{a_3}{5} & \rightarrow \quad a_5 &= \frac{4}{15} \\
a_6 &= \frac{2a_4}{15} & \rightarrow \quad a_6 &= \frac{4}{45} \\
a_7 &= \frac{2a_5}{21} & \rightarrow \quad a_7 &= \frac{8}{315} \\
&\vdots & &\vdots
\end{aligned}$$

So then

$$\begin{aligned}
u(x) &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{8}{315}x^7 + \cdots \\
&= 1 + (2x) + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \frac{1}{24}(2x)^4 + \frac{1}{120}(2x)^5 + \frac{1}{720}(2x)^6 + \frac{1}{5040}(2x)^7 + \cdots \\
&= e^{2x}.
\end{aligned}$$