Exercise 7

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 3 + x^2 - \int_0^x (x-t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this into the integral equation.

$$\begin{aligned}
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots &= 3 + x^2 - \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \cdots)\,dt \\
&= 3 + x^2 - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 - \frac{a_2}{12}x^4 - \frac{a_3}{20}x^5 - \frac{a_4}{30}x^6 - \cdots \\
&= 3 + \left(1 - \frac{a_0}{2}\right)x^2 - \frac{a_1}{6}x^3 - \frac{a_2}{12}x^4 - \frac{a_3}{20}x^5 - \frac{a_4}{30}x^6 - \cdots
\end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned}
a_0 &= 3 \\
a_1 &= 0 \\
a_2 &= 1 - \frac{a_0}{2} & \rightarrow \qquad a_2 &= -\frac{1}{2} \\
a_3 &= -\frac{a_1}{6} & \rightarrow \qquad a_3 &= 0 \\
a_4 &= -\frac{a_2}{12} & \rightarrow \qquad a_4 &= \frac{1}{24} \\
a_5 &= -\frac{a_3}{20} & \rightarrow \qquad a_5 &= 0 \\
a_6 &= -\frac{a_4}{30} & \rightarrow \qquad a_6 &= -\frac{1}{720} \\
&\vdots & &\vdots
\end{aligned}$$

So then

$$\begin{aligned}
u(x) &= 3 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots \\
&= 2 + \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right) \\
&= 2 + \cos x.
\end{aligned}$$