Exercise 11

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$-x + \frac{1}{2}x^2 + \ln(1+x) + x\ln(1+x) = \int_0^x u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Substitute this and the Taylor series expansion of $\ln(1+x)$,

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \cdots,$$

into the integral equation.

$$\begin{aligned} &-x + \frac{1}{2}x^2 + \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \cdots\right) \\ &\qquad + x\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots\right) \\ &\qquad = \int_0^x (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots)\,dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned} &x^2 + \left(\frac{1}{3} - \frac{1}{2}\right)x^3 + \left(-\frac{1}{4} + \frac{1}{3}\right)x^4 + \left(\frac{1}{5} - \frac{1}{4}\right)x^5 + \left(-\frac{1}{6} + \frac{1}{5}\right)x^6 + \left(\frac{1}{7} - \frac{1}{6}\right)x^7 + \cdots \\ &\qquad = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \frac{a_6}{7}x^7 + \cdots \end{aligned}$$

$$\begin{aligned} &x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5 + \frac{1}{30}x^6 - \frac{1}{42}x^7 \\ &\qquad = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \frac{a_6}{7}x^7 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned} &a_0 = 0 && \frac{a_3}{4} = \frac{1}{12} && \frac{a_6}{7} = -\frac{1}{42} && \cdots \\ &\frac{a_1}{2} = 1 && \frac{a_4}{5} = -\frac{1}{20} && && \cdots \\ &\frac{a_2}{3} = -\frac{1}{6} && \frac{a_5}{6} = \frac{1}{30} && && \cdots \end{aligned}$$

So then

$$\begin{aligned} &a_0 = 0 && a_3 = \frac{1}{3} && a_6 = -\frac{1}{6} && \cdots \\ &a_1 = 2 && a_4 = -\frac{1}{4} && && \cdots \\ &a_2 = -\frac{1}{2} && a_5 = \frac{1}{5} && && \cdots \end{aligned}$$

and

$$\begin{aligned}
u(x) &= 2x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots \\
&= x + \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots\right) \\
&= x + \ln(1 + x).
\end{aligned}$$