Exercise 2

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$x\cosh x - \sinh x = \int_0^x u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

Substitute this and the Taylor series expansions of $\cosh x$,

$$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots,$$

and $\sinh x$,

$$\sinh x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \frac{1}{362\,880}x^9 + \cdots,$$

into the integral equation.

$$x\left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots\right) - \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \frac{1}{362\,880}x^9 + \cdots\right) = \int_0^x (a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots)\,dt$$

Simplify the left side and evaluate the integral on the right side.

$$\left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{24} - \frac{1}{120}\right)x^5 + \left(\frac{1}{720} - \frac{1}{5040}\right)x^7 + \left(\frac{1}{40\,320} - \frac{1}{362\,880}\right)x^9 + \cdots = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \frac{a_6}{7}x^7 + \frac{a_7}{8}x^8 + \frac{a_8}{9}x^9 + \cdots$$

$$\frac{1}{3}x^3 + \frac{1}{30}x^5 + \frac{1}{840}x^7 + \frac{1}{45\,360}x^9 + \cdots = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \frac{a_6}{7}x^7 + \frac{a_7}{8}x^8 + \frac{a_8}{9}x^9 + \cdots$$

Match the coefficients of the respective powers of x to determine a_i.

$$a_0 = 0 \qquad \frac{a_3}{4} = 0 \qquad \frac{a_6}{7} = \frac{1}{840} \qquad \cdots$$

$$\frac{a_1}{2} = 0 \qquad \frac{a_4}{5} = \frac{1}{30} \qquad \frac{a_7}{8} = 0 \qquad \cdots$$

$$\frac{a_2}{3} = \frac{1}{3} \qquad \frac{a_5}{6} = 0 \qquad \frac{a_8}{9} = \frac{1}{45\,360} \qquad \cdots$$

So then

$$
\begin{array}{llll}
a_0 = 0 & a_3 = 0 & a_6 = \dfrac{1}{120} & \cdots \\
a_1 = 0 & a_4 = \dfrac{1}{6} & a_7 = 0 & \cdots \\
a_2 = 1 & a_5 = 0 & a_8 = \dfrac{1}{5040} & \cdots
\end{array}
$$

and

$$
\begin{aligned}
u(x) &= x^2 + \frac{1}{6}x^4 + \frac{1}{120}x^6 + \frac{1}{5040}x^8 + \cdots \\
&= x\left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots\right) \\
&= x \sinh x.
\end{aligned}
$$