Exercise 10

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

$$1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 - \sin x - \cos x = \int_0^x (x - t + 1)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\left\{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \sin x - \cos x\right\} = \mathcal{L}\left\{\int_0^x (x - t + 1)u(t)\,dt\right\}$$

Use the fact that the Laplace transform is linear on the left side and apply the convolution theorem on the right side.

$$\begin{aligned}
\mathcal{L}\{1\} + \mathcal{L}\{x\} + \frac{1}{2}\mathcal{L}\{x^2\} + \frac{1}{6}\mathcal{L}\{x^3\} - \mathcal{L}\{\sin x\} - \mathcal{L}\{\cos x\} &= \mathcal{L}\{x+1\}U(s) \\
\frac{1}{s} + \frac{1}{s^2} + \frac{1}{2}\left(\frac{2}{s^3}\right) + \frac{1}{6}\left(\frac{6}{s^4}\right) - \frac{1}{s^2+1} - \frac{s}{s^2+1} &= (\mathcal{L}\{x\} + \mathcal{L}\{1\})U(s) \\
\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} - \frac{1}{s^2+1} - \frac{s}{s^2+1} &= \left(\frac{1}{s^2} + \frac{1}{s}\right)U(s)
\end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned}
\left(\frac{1}{s^2} + \frac{1}{s}\right)U(s) &= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} - \frac{s+1}{s^2+1} \\
(1+s)U(s) &= s + 1 + \frac{1}{s} + \frac{1}{s^2} - \frac{s^3+s^2}{s^2+1} \\
&= (s+1) + \frac{s+1}{s^2} - \frac{s^2(s+1)}{s^2+1} \\
U(s) &= 1 + \frac{1}{s^2} - \frac{s^2}{s^2+1} \\
&= \frac{1}{s^2} + \frac{1}{s^2+1}
\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}
u(x) &= \mathcal{L}^{-1}\{U(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= x + \sin x
\end{aligned}$$