Exercise 6

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

$$\sinh x = \int_0^x e^{x-t} u(t)\, dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\, dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\{\sinh x\} = \mathcal{L}\left\{\int_0^x e^{x-t} u(t)\, dt\right\}$$

Apply the convolution theorem on the right side.

$$\begin{aligned} \mathcal{L}\{\sinh x\} &= \mathcal{L}\{e^x\}U(s) \\ \frac{1}{s^2-1} &= \frac{1}{s-1}U(s) \end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned} \frac{U(s)}{s-1} &= \frac{1}{(s+1)(s-1)} \\ U(s) &= \frac{1}{s+1} \end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned} u(x) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= e^{-x} \end{aligned}$$